MULTIMEDIA UNIVERSITY

STUDENT ID NO

MULTIMEDIA UNIVERSITY

FINAL EXAMINATION

TRIMESTER 2, 2022/23 (T2220)

ERM1136 – DIFFERENTIAL EQUATIONS

(All Sections / Groups)

13th JULY 2023

9.00 AM – 11.00 AM

(2 Hours)

INSTRUCTIONS TO STUDENTS:

1. This exam paper consists of **7 pages** (including cover page) with **4** Questions only.
2. Attempt **all** 4 questions. All questions carry equal marks and the distribution of marks for each question is given.
3. Please print all your answers in the Answer Booklet provided. Show all relevant steps to obtain maximum marks.
4. Several tables are provided in the **Appendix** for your reference.

Question 1 (PD)

(a) The apparent temperature index, A, is a measure of how the temperature feels, and it is based on relative humidity, h, and air temperature, t. Given that

$$A = 0.85t - 22.20h + 1.24th - 0.55,$$

find $\frac{\partial A}{\partial t}$ and $\frac{\partial A}{\partial h}$ when $t = 20$ Fahrenheit and $h = 0.80$.

[4 marks]

(b) Determine the critical points of the function $f(x,y) = y^2 - 2xy + 2x$. Then find the absolute maximum and minimum values of the $f(x,y)$ on the rectangle $D = \{(x,y) | 0 \le x \le 2,\ 0 \le y \le 3\}$.

[14 marks]

(c) Using the method of Lagrange multiplier, determine the minimum and maximum values of $f(x,y) = 2x + y$, subject to the constraint $x^2 + y^2 = 5$.

[7 marks]

Question 2 (ODE)

NOTE: You may use the table in APPENDIX A to solve this question.

(a) Find the general solution of the following non-homogeneous ordinary differential equation:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = x$$

[12 marks]

(b) Using the power series method, find the **first four non-zero terms** of the solution for the following differential equation:

$$y'' + 2xy' + y = 0$$

[13 marks]

Continued...

Question 3 (IT)

NOTE: You may use the tables in APPENDIX B and APPENDIX C to solve this question.

(a) Let two time-domain functions be

$$f(t)=\cos at-\sin at \text{ and } g(t)=t\cosh at,$$

where a is a non-zero constant.

(i) Determine the Laplace transform of $f(t)$. [2 marks]

(ii) Using the Differentiation of Transform property, find $L\{g(t)\}$. [3 marks]

(ii) By applying the Convolution Theorem, show that

$$L\{f(t)*g(t)\}=\frac{1}{(s-a)(s+a)^2},$$

where $*$ denotes convolution. [**Hint**: $x^2-y^2=(x-y)(x+y)$] [3 marks]

(b) (i) Find the inverse Laplace transform of $\frac{1}{s(s-3)}$. Use the Transform of Integral property. [4 marks]

(ii) Show that $L^{-1}\left\{\frac{1}{(s+6)^2}\right\}=te^{-6t}$. [3 marks]

(c) Consider the following ordinary differential equation:

$$y''(t)-8y'(t)+16y(t)=4e^{5t}, \text{ subject to } y(0)=1 \text{ and } y'(0)=8.$$

By taking Laplace transforms, show that

$$L\{y(t)\}=Y(s)=\frac{s-1}{(s-5)(s-4)}.$$

Hence, with the help of partial fractions, determine $y(t)$ via the inverse Laplace transform.

[10 marks]

Continued...

Question 4 (PDE)

The **steady-state temperature**, $u(x,y)$, across a thin rectangular plate is governed by the following boundary value problem (BVP):

$$\begin{aligned} u_{xx}+u_{yy}&=0,\ 0\le x\le a, 0\le y\le b,\\ u(0,y)&=0,\\ u(a,y)&=0,\\ u_y(x,0)&=0,\\ u(x,b)&=K, \end{aligned}$$

where a, b and K are non-zero constants.

(a) Describe the physical set up of the plate: What are its width and height? What are the temperatures at its edges? Which edge is insulated? [**Hint**: It might be helpful to include a diagram in your description.]

[6 marks]

(b) Assuming that the BVP has a separable solution of the form $u(x,y)=X(x)Y(y)$, derive the n-th solution for $X(x)$, i.e. $X_n(x)$.

[12 marks]

(c) If the n-th solution for $Y(y)$ is given by $Y_n(y)=F_n\cosh\left(\dfrac{n\pi y}{a}\right)$, $n=1,2,\ldots$, formulate the **general solution** to the BVP.

[2 marks]

(d) Suppose the right edge of the plate is kept at temperature y^2 instead, identify which of the four boundary conditions (BCs) will have changed. Now, assuming that $u(x,y)=v(x,y)+w(x,y)$, set up the **sub-problems** that solve for $v(x,y)$ and $w(x,y)$. Ensure that each sub-problem deals with only one inhomogeneous BC at a time:

Sub-problem 1	Sub-problem 2
$v_{xx}+v_{yy}=0,\ 0\le x\le a,\ 0\le y\le b$	$w_{xx}+w_{yy}=0, 0\le x\le a,\ 0\le y\le b$
BC 1: ?	BC 1: ?
BC 2: ?	BC 2: ?
BC 3: ?	BC 3: ?
BC 4: ?	BC 4: ?

[5 marks]

Continued...

APPENDIX A:

Rules for the Method of Undetermined Coefficients

Term in $r(x)$	Choice for y_p
ke^{ax}	Ce^{ax}
kx^n $(n=0,1,2,\ldots)$	$C_nx^n+C_{n-1}x^{n-1}+\ldots+C_1x+C_0$
$k\cos wx$, $k\sin wx$	$A\cos wx + B\sin wx$
$ke^{ax}\cos wx$, $ke^{ax}\sin wx$	$e^{ax}(A\cos wx + B\sin wx)$
kx^ne^{ax} $(n=0,1,2,\ldots)$	$(C_nx^n+C_{n-1}x^{n-1}+\ldots+C_1x+C_0)\,e^{ax}$
$kx^n\cos wx$, $kx^n\sin wx$	$(C_nx^n+C_{n-1}x^{n-1}+\ldots+C_1x+C_0)\cos wx +$ $(D_nx^n+D_{n-1}x^{n-1}+\ldots+D_1x+D_0)\sin wx$

Continued...

APPENDIX B:

Common Laplace Transform Pairs

	$f(t)$	$F(s)=L\{f(t)\}$
1.	1	$\frac{1}{s}$
2.	t	$\frac{1}{s^2}$
3.	t^n	$\frac{n!}{s^{n+1}},\ n=1,2,\ldots$
4.	e^{at}	$\frac{1}{s-a}$
5.	$\cos at$	$\frac{s}{s^2+a^2}$
6.	$\sin at$	$\frac{a}{s^2+a^2}$
7.	$\cosh at$	$\frac{s}{s^2-a^2}$
8.	$\sinh at$	$\frac{a}{s^2-a^2}$
9.	$u(t-a)$	$\frac{e^{-as}}{s}$
10.	$\delta(t-a)$	e^{-as}

Continued...

APPENDIX C:

Laplace Transform Properties

	Property Name	Formula
1.	Linearity	$L\{af(t)+bg(t)\} = aL\{f(t)\}+bL\{g(t)\}$
2.	s - shifting	$L\{e^{at}f(t)\} = F(s-a)$
3.	Transform of Derivative	$L\{f'(t)\} = sL\{f(t)\}-f(0)$ $L\{f''(t)\} = s^2L\{f(t)\}-sf(0)-f'(0)$
4.	Transform of Integration	$L\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$
5.	t - shifting	$L\{u(t-a)f(t-a)\} = e^{-as}F(s)$
6.	Differentiation of Transform	$L\{t.f(t)\} = -F'(s)$
7.	Integration of Transform	$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})d\tilde{s}$
8.	Convolution Theorem	$L\{f(t)*g(t)\} = F(s)G(s)$, where $f(t)*g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$.

End of Paper.